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PLANE STRAIN PROBLEM FOR AN INCOMPRESSIBLE NONLINEARLY ELASTIC SOLID

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The plane strain of an incompressible body is studied with geometrical and physical nonlinearity and potential forces taken into account. A nonlinear system of equations for strains is obtained in actual variables, and conditions of its ellipticity are derived in terms of the elastic potential. Boundary conditions for strains are found from specified loads. Analytical solutions of the boundary problem in strains and their corresponding stress fields are found for the case of identical elongations.

Key words: *stresses, strains, elongations, potential forces, incompressibility, nonlinearity, ellipticity.*

Many important elastic problems cannot be solved with sufficient accuracy in terms of linear theory. To study these problems, it is necessary to reject simplifying assumptions and use nonlinear elastic models which allow a more accurate accounting for material properties and strain behavior. However, accounting for nonlinearity considerably complicates the study; therefore, for the solution of some classes of nonlinear problems important for applications, it is common to use some simplifications. Among such problems is the plane strain problem.

In this paper, the plane problem is considered in actual variables for an incompressible homogeneous cylindrical solid which is in equilibrium under the action of surface and potential body forces.

The study is performed using a nonlinear elastic model which includes equilibrium equations, the incompressibility condition, the representation of the elastic potential in terms of strain invariants and an expression of the invariants in terms of its components, the Murnaghan law relating the Cauchy stresses and Almansi strains, and the expression of strains in terms of displacements [1, 2]. In actual Cartesian variables x_1, x_2, and x_3, the equilibrium equations and the boundary conditions in forces are written as

$$\frac{\partial P_{kl}}{\partial x_l} - \frac{\partial V}{\partial x_k} = 0, \qquad E_1 - 2E_2 + 4E_3 = 0, \qquad U = U(E_1, E_2, E_3),$$

$$E_1 = E_{nn}, \qquad 2E_2 = E_{mm}E_{nn} - E_{mn}E_{nm}, \qquad E_3 = |E_{kl}|,$$

$$P_{kl} = -q_0\delta_{kl} + (\delta_{kn} - 2E_{kn})\,\frac{\partial U}{\partial E_{ln}}, \qquad 2E_{kl} = \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} - \frac{\partial u_n}{\partial x_k}\,\frac{\partial u_n}{\partial x_l}, \tag{1}$$

$$P_{kl}n_l\Big|_{S_*} = p_k.$$

Here U and V are the elastic and force potentials, q_0 is the Lagrangian multiplier, u_k, p_k, and n_k are the components of the displacement, load, and the normal to the surface of the solid, P_{kl} and E_{kl} are the components of the symmetric stress and strain tensors, δ_{kl} is the Kronecker symbol, E_1, E_2, and E_3 are the basic strain invariants, and S_* is the surface of the deformed solid; the indices take values 1, 2, and 3; the summation is performed over repeated indices. In relations (1), the force and elastic potentials, the load, and the surface of the solid are assumed to be specified and the remaining quantities are to be determined.

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Let a cylindrical body of cross-sectional area S with contour L be subjected to plane deformation. The displacement and potential body forces are parallel to the strain plane. The load density is specified on the lateral surface of the body, and the longitudinal component of the resultant load at the ends of the body. The displacement, forces, and load do not vary along the generatrix of the cylinder. In the coordinates x_1, x_2, x_3 ($x_1 = x$ and $x_2 = y$ are transverse coordinates, and $x_3 = z$ is the longitudinal coordinate), these conditions have the form

$$u_1 = u(x,y), \qquad u_2 = v(x,y), \qquad u_3 = 0, \qquad V = V(x,y),$$
$$p_k\Big|_L = p_k(x,y), \qquad P_3 = \int\limits_S p_3\,dS. \tag{2}$$

In the case of plane strain, relation (1) contains only three nonzero strain components (the matrix of the strain components is two-dimensional). These components depend nonlinearly on displacement gradients (geometrical nonlinearity) and are functions of the transverse coordinates:

$$1 - 2E_{11} = \Big(1 - \frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2, \qquad 1 - 2E_{22} = \Big(1 - \frac{\partial v}{\partial y}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2, \qquad E_{33} = 0,$$
$$2E_{12} = \frac{\partial u}{\partial y}\Big(1 - \frac{\partial u}{\partial x}\Big) + \frac{\partial v}{\partial x}\Big(1 - \frac{\partial v}{\partial y}\Big), \qquad E_{23} = E_{31} = 0, \qquad E_{kl} = E_{kl}(x,y). \tag{3}$$

The strain invariants do not vary along the generatrix and, due to the incompressibility condition, are linked by the relations

$$E = E_1 = E_{11} + E_{22}, \qquad E_2 = E_{11}E_{22} - E_{12}^2,$$
$$E_3 = 0, \qquad E_k = E_k(x,y), \qquad E_1 - 2E_2 = 0. \tag{4}$$

From relations (4), it follows that the strain invariants and the elastic potential are functions of the first invariant

$$E_1 = E, \qquad 2E_2 = E, \qquad E_3 = 0, \qquad U(E_1, E_2, E_3) = U(E).$$

The first strain invariant expressed in displacements and transformed in view of the incompressibility condition written in displacements

$$\Big(1 - \frac{\partial u}{\partial x}\Big)\Big(1 - \frac{\partial v}{\partial y}\Big) - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x} = 1,$$

admits the representation

$$2E = -\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)^2 - \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2,$$

which implies that, for plane deformation, this invariant is a negative quantity.

In the case considered, the elastic-potential gradient along the strain tensor is a spherical tensor:

$$E = E_{ln}\delta_{nl}, \qquad \frac{\partial E}{\partial E_{ln}} = \delta_{nl}, \qquad \frac{\partial U}{\partial E_{ln}} = U'\delta_{nl} \qquad \Big(U' = \frac{dU}{dE}\Big).$$

As a result, the Murnaghan law reduces to a quasilinear dependence of the stress on the strain (physical nonlinearity) and the pressure q:

$$P_{kl} = -q\delta_{kl} - 2U'E_{kl}, \qquad q = q_0 - U', \qquad U' = U'(E), \qquad E = E_{11} + E_{22}.$$

This law can be represented in expanded form

$$P_{11} = -q - 2U'E_{11}, \qquad P_{22} = -q - 2U'E_{22}, \qquad P_{33} = -q,$$
$$P_{12} = -2U'E_{12}, \qquad P_{23} = 0, \qquad P_{31} = 0. \tag{5}$$

From this it follows that, in contrast to the strains, the matrix of the stress components is three-dimensional.

In view of relation (5), the equilibrium equations in (1) become

$$\frac{\partial(P_{11}-V)}{\partial x}+\frac{\partial P_{12}}{\partial y}=0,\qquad \frac{\partial(P_{22}-V)}{\partial y}+\frac{\partial P_{12}}{\partial x}=0,\qquad \frac{\partial P_{33}}{\partial z}=0. \tag{6}$$

The last equation in (6) implies that, similarly to the strains, the pressure does not vary along the generatrix. Hence, the stresses (5) are functions of the transverse coordinates: $P_{kl}=P_{kl}(x,y)$.

On the lateral surface of the cylinder, the normal is orthogonal to the generatrix: $(n_k)=(n_1,n_2,0)$. From this and from the stress relations (5), it follows that the lateral load is completely determined by its value on the cross-sectional contour and belongs to the cross-sectional plane:

$$P_{11}n_1+P_{12}n_2\Big|_L=p_1,\qquad P_{12}n_1+P_{22}n_2\Big|_L=p_2,\qquad 0=p_3. \tag{7}$$

At the end of the cylinder $S_\pm$ (the plus and minus signs correspond to the upper and lower ends), the normal is parallel to the generatrix: $(n_k)=(0,0,\pm1)$ and the longitudinal component of the resultant end load (2) is equal to

$$p_1^\pm=p_2^\pm=0,\qquad p_3^\pm=\pm P_{33},\qquad P_3=\pm\int\limits_S P_{33}\,dS. \tag{8}$$

Relations (3)–(8) define the plane problem of nonlinear elasticity and allow the formulation of the boundary-value problem for strains. Unlike in the boundary-value problem for stresses, in this case, there is no need for the inversion of the nonlinear Murnaghan law.

The incompressibility condition (4) defines the final relations between the strain components:

$$(1-2E_{11})(1-2E_{22})-(2E_{12})^2=1. \tag{9}$$

Substitution of the stresses (5) into the equilibrium equations (6) yields the following differential equations for strains and pressure:

$$\begin{gathered}\frac{\partial(q+V)}{\partial x}=-\frac{\partial(2U'E_{11})}{\partial x}-\frac{\partial(2U'E_{12})}{\partial y},\qquad \frac{\partial(q+V)}{\partial y}=-\frac{\partial(2U'E_{22})}{\partial y}-\frac{\partial(2U'E_{12})}{\partial x},\\ \frac{\partial q}{\partial z}=0,\qquad U'=U'(E_{11}+E_{22}),\end{gathered} \tag{10}$$

and substitution of (5) into boundary equalities (6) gives boundary conditions for these quantities:

$$-qn_1-2U'(E_{11}n_1+E_{12}n_2)\Big|_L=p_1,\qquad -qn_2-2U'(E_{21}n_1+E_{22}n_2)\Big|_L=p_2. \tag{11}$$

Eliminating the pressure from the first and second equations in (10) (for this, we differentiate the first of them with respect to y and the second with respect to x and subtract the results), we obtain an equation only for strains:

$$\frac{\partial}{\partial y}\Big(\frac{\partial\,(2U'E_{11})}{\partial x}+\frac{\partial\,(2U'E_{12})}{\partial y}\Big)=\frac{\partial}{\partial x}\Big(\frac{\partial\,(2U'E_{22})}{\partial y}+\frac{\partial\,(2U'E_{12})}{\partial x}\Big). \tag{12}$$

After integration, the third equation in (10) defines the pressure as a function of the transverse coordinates $q=q(x,y)$. According to the first and second equations, this function is established from known strains and elastic potential using the quadrature:

$$\begin{gathered}q+V=\int\Big(\frac{\partial\,(q+V)}{\partial x}\,dx+\frac{\partial\,(q+V)}{\partial y}\,dy\Big)=-W(x,y)+B,\qquad B=\mathrm{const},\\ W=\int\Big(\frac{\partial\,(2U'E_{11})}{\partial x}+\frac{\partial\,(2U'E_{12})}{\partial y}\Big)\,dx+\Big(\frac{\partial\,(2U'E_{22})}{\partial y}+\frac{\partial\,(2U'E_{12})}{\partial x}\Big)\,dy.\end{gathered} \tag{13}$$

By virtue of Eq. (12), the integral in formulas (13) does not depend on the integration path, and the integration constant can be calculated by the integral condition in (8) and relations (5) and (13):

$$B=\frac{1}{S}\Big(\int\limits_S(V+W)\,dS-P_3\Big). \tag{14}$$

In particular, for $P_3 = 0$, this constant is the value of the sum of the functions $V + W$ averaged over the cross section of the solid.

A closed system of equations for strains is obtained if relations (8) and (12) are supplemented by the strain compatibility equation that follows from the strain–displacement relations (3) of after elimination of displacements. For this, we calculate the first and second derivatives of strains with respect to coordinates. Then, the first derivatives

$$\frac{\partial E_{11}}{\partial x}, \quad \frac{\partial E_{11}}{\partial y}, \quad \frac{\partial E_{22}}{\partial x}, \quad \frac{\partial E_{22}}{\partial y}, \quad \frac{\partial E_{12}}{\partial x}, \quad \frac{\partial E_{12}}{\partial y} \tag{15}$$

are functions of the first and the second derivatives of the displacements, and the second derivatives

$$\frac{\partial^2 E_{11}}{\partial y^2}, \qquad \frac{\partial^2 E_{22}}{\partial x^2}, \qquad \frac{\partial^2 E_{12}}{\partial x\, \partial y} \tag{16}$$

depend on the derivatives of the displacements of the first–third order. From expressions for quantities (16), the third derivatives can be eliminated using the following combination:

$$\frac{\partial^2 E_{11}}{\partial y^2} + \frac{\partial^2 E_{22}}{\partial x^2} - 2\,\frac{\partial^2 E_{12}}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2} - \Big(\frac{\partial^2 u}{\partial x\, \partial y}\Big)^2 - \Big(\frac{\partial^2 v}{\partial x\, \partial y}\Big)^2. \tag{17}$$

Determining the second derivatives of displacements in terms of strains and inserting them into equality (17), we obtain a relation that contains only strains and the first derivatives of displacements. The latter quantities are included in this relation tin the form of the combinations presented in (3), and, hence, they can be eliminated by means of these formulas. As a result, we obtain a nonlinear equation of strain compatibility that contains strains and their first and second derivatives with respect to the coordinates:

$$\begin{aligned}\frac{\partial^2 E_{11}}{\partial y^2} + \frac{\partial^2 E_{22}}{\partial x^2} - 2\,\frac{\partial^2 E_{12}}{\partial x\, \partial y} &= (1 - 2E_{11})\Big[\frac{\partial E_{22}}{\partial y}\Big(2\,\frac{\partial E_{12}}{\partial x} - \frac{\partial E_{11}}{\partial y}\Big) - \Big(\frac{\partial E_{22}}{\partial x}\Big)^2\Big] \\ &+ (1 - 2E_{22})\Big[\frac{\partial E_{11}}{\partial x}\Big(2\,\frac{\partial E_{12}}{\partial y} - \frac{\partial E_{22}}{\partial x}\Big) - \Big(\frac{\partial E_{11}}{\partial y}\Big)^2\Big] \\ &+ 2E_{12}\Big[\frac{\partial E_{11}}{\partial x}\,\frac{\partial E_{22}}{\partial y} + \Big(2\,\frac{\partial E_{12}}{\partial x} - \frac{\partial E_{11}}{\partial y}\Big)\Big(2\,\frac{\partial E_{12}}{\partial y} - \frac{\partial E_{22}}{\partial x}\Big) - 2\,\frac{\partial E_{11}}{\partial y}\,\frac{\partial E_{22}}{\partial x}\Big].\end{aligned} \tag{18}$$

System (9), (12), (18) and boundary conditions (11) transformed by using (13)

$$\begin{aligned}(V + W - B)n_1 - 2U'(E_{11}n_1 + E_{12}n_2)\Big|_L &= p_1, \\ (V + W - B)n_2 - 2U'(E_{12}n_1 + E_{22}n_2)\Big|_L &= p_2\end{aligned} \tag{19}$$

[W and B are defined in (13)] constitute a nonlinear boundary-value problem for strains. Given the strains, the pressure is defined by formula (13) and the stresses by equalities (5).

To determine the type of system (9), (12), (18), we transform from strains to the quantities f, g, and h:

$$f = 1 - 2E_{11}, \qquad g = 2E_{12}, \qquad h = 1 - 2E_{22}. \tag{20}$$

From formulas (3), it follows that the quantities f and h are positive:

$$f > 0, \qquad h > 0. \tag{21}$$

Then, the strain relations can be written as

$$\begin{aligned}& fh - g^2 = 1, \qquad \frac{\partial^2 U'(h - f)}{\partial x\, \partial y} + \frac{\partial^2 U' g}{\partial y^2} - \frac{\partial^2 U' g}{\partial x^2} = 0, \qquad U' = U'(E), \\ & 2\Big(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 2\,\frac{\partial^2 g}{\partial x\, \partial y}\Big) = g\Big[2\,\frac{\partial f}{\partial y}\,\frac{\partial h}{\partial x} - \frac{\partial f}{\partial x}\,\frac{\partial h}{\partial y} - \Big(2\,\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\Big)\Big(2\,\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\Big)\Big] \\ & + f\Big[\frac{\partial h}{\partial y}\Big(2\,\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\Big) + \Big(\frac{\partial h}{\partial x}\Big)^2\Big] + h\Big[\frac{\partial f}{\partial x}\Big(2\,\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\Big) + \Big(\frac{\partial f}{\partial y}\Big)^2\Big], \qquad E = 1 - \frac{f + h}{2},\end{aligned} \tag{22}$$

boundary conditions (19) become

$$p_1 = (V - N - B)n_1 + U'(fn_1 - gn_2)\Big|_L, \qquad p_2 = (V - N - B)n_2 + U'(hn_2 - gn_1)\Big|_L,$$

$$W = U' - N, \qquad B = \frac{1}{S}\Big(\int (V + U' - N)\, dS - P_3\Big), \tag{23}$$

$$N = \int \Big(\frac{\partial U' f}{\partial x} - \frac{\partial U' g}{\partial y}\Big)\, dx + \Big(\frac{\partial U' h}{\partial y} - \frac{\partial U' g}{\partial x}\Big)\, dy,$$

and for the pressure and stresses, we obtain the expressions

$$q = B + N - V - U'; \tag{24}$$

$$P_{11} = V - N - B + U'f, \qquad P_{22} = V - N - B + U'h,$$

$$P_{33} = V + U' - N - B, \qquad P_{12} = -U'g, \qquad P_{23} = 0, \qquad P_{31} = 0. \tag{25}$$

Using the first equation of system (22), we obtain

$$g = \sqrt{fh - 1}. \tag{26}$$

Substituting (26) into the second and third equations (22), we write them in expanded form

$$G_1 = [2U' - U''(h - f)]\, \frac{\partial^2 h}{\partial x\, \partial y} - [2U' + U''(h - f)]\, \frac{\partial^2 f}{\partial x\, \partial y}$$

$$+ \Big(U' \frac{f}{g} - U'' g\Big)\Big(\frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial x^2}\Big) + \Big(U' \frac{h}{g} - U'' g\Big)\Big(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\Big) + R = 0, \tag{27}$$

$$G_2 = g\Big(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 h}{\partial x^2}\Big) + h\, \frac{\partial^2 f}{\partial x\, \partial y} + f\, \frac{\partial^2 h}{\partial x\, \partial y} + T = 0.$$

Here the function g is expressed in terms of the quantities f and h by formula (26), and R and T denote the terms not containing higher (second) derivatives of the required functions.

To system (27) there corresponds the second-order characteristic matrix [3]

$$(A_{kl}) = \Big(\sum_{m+n=2} \Big\{\partial G_k \Big/ \partial\Big(\frac{\partial^2 g_l}{\partial x_m\, \partial x_n}\Big)\Big\} s^m t^n\Big), \qquad (g_l) = (f, h)$$

with the elements

$$A_{11} = ((h/g)U' - gU'')(t^2 - s^2) - [2U' + (h - f)U'']st, \qquad A_{22} = gs^2 + fst,$$

$$A_{12} = ((f/g)U' - gU'')(t^2 - s^2) + [2U' - (h - f)U'']st, \qquad A_{21} = hst + gt^2.$$

The characteristic determinant $A = |A_{kl}|$ of system (27) is a fourth-degree polynomial in the variables s and t:

$$A = A_{11}A_{22} - A_{12}A_{21} = -U'(s^2 + t^2)(hs^2 + 2gst + ft^2) + U''[g(s^2 - t^2) + (f - h)st]^2. \tag{28}$$

From inequalities (21) and the incompressibility conditions (23), it follows that the quantities f, h, and g obey the conditions

$$f > 0, \qquad h > 0, \qquad fh - g^2 > 0,$$

because of which the quadratic form presented to (28) is positive definite according to the Sylvester criterion:

$$hs^2 + 2gst + ft^2 > 0. \tag{29}$$

Relations (28) and (29) imply that if the second derivative of the elastic potential is opposite in sign to the first derivative or is equal to zero, the characteristic determinant is different from zero:

$$A > 0 \quad \text{at} \quad U' < 0, \quad U'' \geqslant 0; \qquad A < 0 \quad \text{at} \quad U' > 0, \quad U'' \leqslant 0. \tag{30}$$

With the conditions on the elastic potential (30), the characteristic equation $A = 0$ has no real roots, and, hence, the nonlinear system (27) is an elliptic system. Thus, the type of the system of equations for strains is determined by the form of the elastic potential.

The ellipticity conditions are satisfied, in particular, for the Rivlin–Saunders quadratic potential [4], which generalizes the Mooney linear potential and models large elastic strains of incompressible rubber-like materials:

$$U = aE^2 - bE + c \qquad (a > 0, \quad b > 0, \quad c > 0, \quad E < 0)$$

(a, b, and c are elastic constants of the material).

The incompressibility condition $fh - g^2 = 1$ allows some general conclusions concerning the nature of the deformation defined by the nonlinear elongations E_{11} and E_{22} and the shear E_{12}. In particular, from this condition, it follows that plane deformation of an incompressible body only with shears or only with elongations is impossible. Indeed, in the absence of elongations ($E_{11} = E_{22} = 0$), we have $f = h = 1$, and the incompressibility condition reduces to the absence of shears: $g = 0$ and $E_{12} = 0$, i.e., strains should be absent. In the absence of shears ($E_{12} = 0$ and $g = 0$), the incompressibility condition becomes $fh = 1$, which contradicts the properties (21) of the functions f and h: each of these functions, being positive, can be smaller than unity. Hence, this deformation is also impossible. At the same time, deformation with shears and identical elongations ($E_{12} \neq 0$ and $E_{11} = E_{22}$) is admitted by the incompressibility condition. For this case, we find the form of strains and their corresponding load. In terms of quantities (20), the specified conditions are written as

$$f = h, \qquad h^2 - g^2 = 1. \tag{31}$$

Relation (31) allow the required quantities to be expressed in terms of one of them, for example g. The independent invariant of the strain and the elastic potential are also determined through this quantity:

$$f = h = \sqrt{1 + g^2}\,, \qquad E = 1 - h = 1 - \sqrt{1 + g^2}\,, \qquad U'(E) = U'(g). \tag{32}$$

If conditions (32) are satisfied, the following relations hold:

$$\frac{\partial^2 U'(h - f)}{\partial x \partial y} = 0,$$

$$g\Big[2\,\frac{\partial f}{\partial y}\,\frac{\partial h}{\partial x} - \frac{\partial f}{\partial x}\,\frac{\partial h}{\partial y} - \Big(2\,\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\Big)\Big(2\,\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\Big)\Big]$$

$$+ f\Big[\frac{\partial h}{\partial y}\Big(2\,\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\Big) + \Big(\frac{\partial h}{\partial x}\Big)^2\Big] + h\Big[\frac{\partial f}{\partial x}\Big(2\,\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\Big) + \Big(\frac{\partial f}{\partial y}\Big)^2\Big]$$

$$= \Big(\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\Big)\,\frac{\partial\,(h^2 - g^2)}{\partial x} + \Big(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\Big)\,\frac{\partial\,(h^2 - g^2)}{\partial y} = 0,$$

by virtue of the aforesaid, the differential equalities in (22) are simplified and, in view of (32), become equations for the quantity $g(x, y)$:

$$\frac{\partial^2 gU'(g)}{\partial y^2} - \frac{\partial\, gU'(g)}{\partial x^2} = 0, \qquad \frac{\partial^2 \sqrt{1 + g^2}}{\partial x^2} + \frac{\partial^2 \sqrt{1 + g^2}}{\partial y^2} + 2\,\frac{\partial^2 g}{\partial x\,\partial y} = 0. \tag{33}$$

In this case, we have

$$N = U'\sqrt{1 + g^2} - H(x, y), \qquad H = \int \frac{\partial\, gU'}{\partial y}\,dx + \frac{\partial\, gU'}{\partial x}\,dy. \tag{34}$$

The expression for the pressure (24) becomes

$$\begin{gathered} q = B + (\sqrt{1 + g^2} - 1)U' - H - V, \\ B = \frac{1}{S}\Big(\int \Big(H + V - \Big(\sqrt{1 + g^2} - 1\Big)U'\Big)\,dS - P_3\Big), \end{gathered} \tag{35}$$

the contour load (23) is equal to

$$p_1 = (H + V - B)n_1 - gU'n_2\Big|_L, \qquad p_2 = (H + V - B)n_2 - gU'n_1\Big|_L, \tag{36}$$

and the stress field (25) is given by

$$P_{11} = P_{22} = H + V - B, \qquad P_{33} = H + V - \left(\sqrt{1+g^2} - 1\right)U' - B,$$
$$P_{12} = -gU', \qquad P_{23} = 0, \qquad P_{31} = 0. \tag{37}$$

The solution of the nonlinear system (33) is sought in the form of a function of one argument $g = g(s)$ ($s = x + y$). Then, differentiation with respect to the coordinates reduces to differentiation with respect to this argument:

$$\frac{\partial}{\partial x} = \frac{d}{ds}\frac{\partial s}{\partial x} = \frac{d}{ds}, \qquad \frac{\partial}{\partial y} = \frac{d}{ds}\frac{\partial s}{\partial y} = \frac{d}{ds}.$$

In this case, the first equation in (33) is identically satisfied for any form of the elastic potential:

$$\frac{\partial^2 gU'}{\partial y^2} - \frac{\partial^2 gU'}{\partial x^2} = \frac{d^2 gU'}{ds^2} - \frac{d^2 gU'}{ds^2} = 0,$$

and the second determines the form of the required function. Indeed, this equation admits the representation

$$\frac{d^2}{ds^2}\left(\sqrt{1+g^2} + g\right) = 0$$

and, after integration, it gives the relation

$$\sqrt{1+g^2} + g = ms + n, \qquad m = \text{const}, \qquad n = \text{const}.$$

From this it follows that required function and the quantities from expressions (32) contain two free parameters:

$$g = \frac{1}{2}\left(ms + n - \frac{1}{ms+n}\right), \qquad f = h = \frac{1}{2}\left(ms + n + \frac{1}{ms+n}\right). \tag{38}$$

According to (34) and (35), the function H and the constant B have the values

$$H = \int \frac{d\,gU'}{ds}\,d(x+y) = gU', \qquad B = \frac{1}{S}\left(\int (V - (h-g-1)U')\,dS - P_3\right).$$

In view of (38), the expressions for the pressure (35) and stresses (37) become

$$q = B - V + (h-g-1)U',$$
$$P_{11} = P_{22} = gU' + V - B, \qquad P_{33} = V - (h-g-1)U' - B, \tag{39}$$
$$P_{12} = -gU', \qquad P_{23} = 0, \qquad P_{31} = 0,$$

and their corresponding contour load (36) is given by the relations

$$p_1 = (V-B)n_1 + gU'(n_1 - n_2)\Big|_L, \qquad p_2 = (V-B)n_2 - gU'(n_1 - n_2)\Big|_L. \tag{40}$$

Another solution of system (33) is defined as the function $g = g(t)$ ($t = x - y$). Then, the derivatives with respect to the coordinates differ only in sign:

$$\frac{\partial}{\partial x} = \frac{d}{dt}\frac{\partial t}{\partial x} = \frac{d}{dt}, \qquad \frac{\partial}{\partial y} = \frac{d}{dt}\frac{\partial t}{\partial y} = -\frac{d}{dt}.$$

In system (33), the first equation is identically satisfied for any elastic potential:

$$\frac{\partial^2 gU'}{\partial y^2} - \frac{\partial^2 gU'}{\partial x^2} = \frac{d^2 gU'}{dt^2} - \frac{d^2 gU'}{dt^2} = 0,$$

and the second equation becomes

$$\frac{d^2}{dt^2}\left(\sqrt{1+g^2} - g\right) = 0$$

and defines the solution with two arbitrary constants

$$\sqrt{1+g^2} - g = kt + l, \qquad k = \text{const}, \qquad l = \text{const}.$$

From this and from equality (32), the required quantities are expressed as

$$g = \frac{1}{2}\Big(\frac{1}{kt+l} - kt - l\Big), \qquad f = h = \frac{1}{2}\Big(\frac{1}{kt+l} + kt + l\Big). \tag{41}$$

In view of the relations

$$H = -gU', \qquad B = \frac{1}{S}\Big(\int (V - (h+g-1)U')\,dS - P_3\Big),$$

the pressure

$$q = B - V + U'(h+g-1), \tag{42}$$

stresses

$$\begin{gathered} P_{11} = P_{22} = V - gU' - B, \qquad P_{33} = V - (h+g-1)U' - B, \\ P_{12} = -gU', \qquad P_{23} = 0, \qquad P_{31} = 0, \end{gathered} \tag{43}$$

and contour load

$$p_1 = (V-B)n_1 - gU'(n_1+n_2)\Big|_L, \qquad p_2 = (V-B)n_2 - gU'(n_1+n_2)\Big|_L \tag{44}$$

correspond to the quantities (41). To concretize the expressions for the force quantities (39), (40), and (42)–(44), it is sufficient to take definite force and elastic potentials, longitudinal end force, and cylinder cross-sectional shape.

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